Let $\Omega = B_1(0) \subset \mathbb{R}^2$. Given $g \in C(\overline{\Omega})$, we define

$$u(x) = \frac{1-|x|^2}{2\pi} \int_{\partial\Omega} \frac{g(\sigma)}{|x-\sigma|^2} d\sigma. \tag{1}$$

Theorem 1. *In Ω, $u(x)$ is smooth and $\Delta u(x) = 0$ holds.*

Proof. Since $|x - \sigma| > 0$ for $x \in \Omega$, $|x - \sigma|^{-2}$ is smooth in Ω. Hence, $u(x)$ is smooth in Ω. Next, we recall the Green function

$$G(x, y) = -\tfrac{1}{2\pi} \left(\log|x-y| - \log\left(|x||y - \tfrac{x}{|x|^2}|\right)\right), \tag{2}$$

where $x^* = x|x|^{-2}$, and we calculate

$$\nabla_y G(x, y) = \frac{1}{2\pi}\left(\frac{x-y}{|x-y|^2} - \frac{x^*-y}{|x^*-y|^2}\right). \tag{3}$$

If $\sigma \in \partial B_1(0)$, then we have $|x^* - \sigma| = |x|^{-1}|x - \sigma|$. Hence,

$$\nabla_y G(x, \sigma) = \frac{1}{2\pi}\left(\frac{x-\sigma}{|x-\sigma|^2} - \frac{|x|^2(x^*-\sigma)}{|x-\sigma|^2}\right) = \frac{\sigma(1-|x|^2)}{2\pi|x-\sigma|^2}. \tag{4}$$

Since $\nu(\sigma) = \sigma$ on $\partial B_1(0)$, we have

$$u(x) = \int_{\partial\Omega} G_\nu(x, \sigma) g(\sigma) d\sigma, \tag{5}$$

which is harmonic in Ω. □

Theorem 2. *$u \in C(\overline{\Omega})$ and $u = g$ on $\partial\Omega$.*

Proof. We define

$$K(x, y) = \frac{1-|x|^2}{2\pi|x-y|^2}, \tag{6}$$

and express u by

$$u(x) = \int_{\partial B_1(0)} g(\sigma) K(x, \sigma) d\sigma. \tag{7}$$

We notice that if $g(x) = 1$ is a constant function then by the uniqueness $u(x) = 1$. Therefore,

$$1 = \int_{\partial\Omega} K(x, \sigma) d\sigma. \tag{8}$$

Given $\sigma_0 \in \partial\Omega$, we multiply $g(\sigma_0)$ to (8) and subtract from (8) to obtain

$$u(x) - g(\sigma_0) = \int_{\partial\Omega} K(x,\sigma)(g(\sigma) - g(\sigma_0))d\sigma. \tag{9}$$

Since $g \in C(\overline{\Omega})$, given $\sigma_0 \in \partial B_1(0)$ and $\epsilon > 0$, there exists $\delta > 0$ such that $|g(x) - g(\sigma_0)| \leqslant \frac{\epsilon}{2}$ holds if $|x - \sigma_0| \leqslant \delta$. Hence, (9) implies

$$|u(x) - g(\sigma_0)| \leqslant \int_{\partial\Omega\cap B_\delta(\sigma_0)} K(x,\sigma)\,|g(\sigma) - g(\sigma_0)|\,d\sigma \tag{10}$$

$$+ \int_{\partial\Omega\setminus B_\delta(\sigma_0)} K(x,\sigma)\,|g(\sigma) - g(\sigma_0)|\,d\sigma \tag{11}$$

$$\leqslant \frac{\epsilon}{2} + 2\max|g| \int_{\partial\Omega\setminus B_\delta(\sigma_0)} K(x,\sigma)d\sigma. \tag{12}$$

If $|x - \sigma_0| \leqslant \frac{\delta}{2}$, then $|x - \sigma| \geqslant \frac{\delta}{2}$ holds for $\sigma \notin B_\delta(\sigma_0)$. Thus,

$$|u(x) - g(\sigma_0)| \leqslant \frac{\epsilon}{2} + \frac{2\max|g|}{(\delta/2)^2}(1 - |x|^2). \tag{13}$$

Moreover,

$$\left|1 - |x|^2\right| = \left||\sigma_0|^2 - |x|^2\right| = |(\sigma_0 - x)(\sigma_0 + x)| \leqslant 2|\sigma_0 - x|. \tag{14}$$

Therefore, there exists some $\delta' \leqslant \frac{\delta}{2}$ such that if $|\sigma_0 - x| \leqslant \delta'$ then $|u(x) - g(\sigma_0)| \leqslant \epsilon$, namely u is continuous at $\sigma_0 \in \partial\Omega$ and $u(\sigma_0) = g(\sigma_0)$. □